Linear Programming Relaxation Approach to Weighted CSP

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Outline

- Notation
- Part 1: LP relaxation approach of WCSP, following [Schlesinger-1976]
- Part 2: Algorithms to solve the LP relaxation
- Part 3: Higher order relaxations

Notation

V	(finite) set of variables, totally ordered
$v \in V$	a single variable
X_v	(finite) domain of variable $v \in V$
$x_v \in X_v$	state of variable $v \in V$
$A \subseteq V$	a subset of variables
$X_A = \prod_{v \in A} X_v$	joint domain of variables $A \subseteq V$ (ordered by the order on V)
$x_A \in X_A$	joint state ('tuple', 'configuration') of variables $A \subseteq V$

Convention: "Implicit restriction"

For $B \subset A$, if symbols x_A and x_B appear in the same logical expression, x_B denotes the restriction of joint state x_A onto variables B.

$\bar{\mathbb{R}}$	extended reals, $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$
$f_A \colon X_A \to \bar{\mathbb{R}}$	constraint with scope $A \subseteq V$
2^V	the set of all subsets of V
$\binom{V}{k}$	the set of all k-element subsets of V

Definition (Constraint network)

Let $E \subseteq 2^V$ be a hypergraph. Let each hyperedge $A \in E$ be assigned a constraint $f_A\colon X_A \to \bar{\mathbb{R}}$. This collection of constraints is called a constraint network.

Denoting $T(E) = \{\, (A, x_A) \mid A \in E,\ x_A \in X_A \,\}$, a constraint network is a mapping

$$\begin{aligned} f : \quad T(E) &\to \bar{\mathbb{R}} \\ (A, x_A) &\mapsto f_A(x_A) \end{aligned}$$

Definition (Weighted CSP, WCSP)

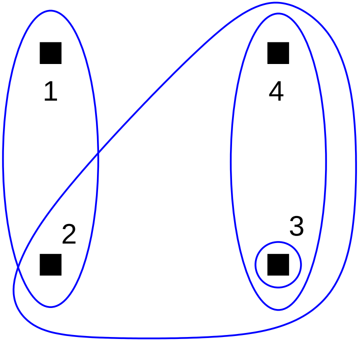
Given a constraint network f, compute the value (and, optionally, a maximiser) of

$$\max_{x_V \in X_V} \sum_{A \in E} f_A(x_A)$$

Example: A ternary WCSP

Let $V = (1, 2, 3, 4)$ and $E = \{(2, 3, 4), (1, 2), (3, 4), (3)\}$. Then

$$\max_{x_V} \sum_{A \in E} f_A(x_A) = \max_{x_1, x_2, x_3, x_4} [f_{234}(x_2, x_3, x_4) + f_{12}(x_1, x_2) + f_{34}(x_3, x_4) + f_3(x_3)]$$



Example: A binary WCSP

Let $E = \binom{V}{1} \cup E'$ with $E' \subseteq \binom{V}{2}$ (all unary and some binary constraints)

$$\max_{x_V} \sum_{A \in E} f_A(x_A) = \max_{x_V} \Big[\sum_{v \in V} f_v(x_v) + \sum_{vv' \in E'} f_{vv'}(x_v, x_{v'}) \Big]$$

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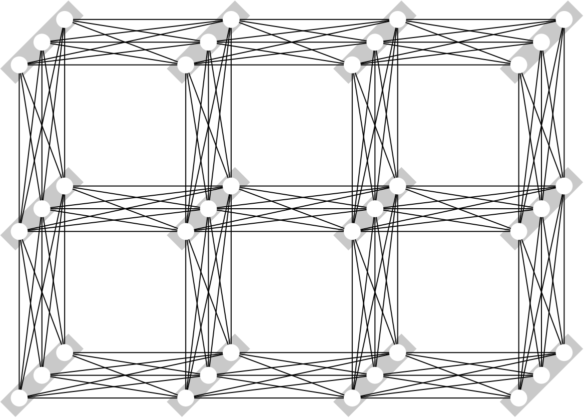
Microstructure for E a grid graph and $X_v = \{1, 2, 3\}$:



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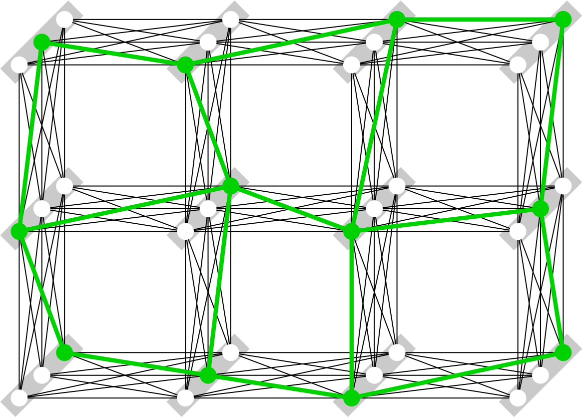

Part 1

LP Relaxation of WCSP by [Schlesinger-1976]
(more precisely, its n-ary generalisation)

Outline of Part 1

- Formulate two LPs, each yielding an upper bound on WCSP:
  - The first LP is a continuous relaxation of an integer LP formulation of WCSP
  - The second LP minimises an upper bound on WCSP by equivalent transformations
- Show that the two LPs are dual to each other
- Characterise when the upper bound is tight or minimal

Note: LP relaxation of WCSP yielding the same bound as [Schlesinger-1976] was proposed also by other researchers [Koster-1998, Chekuri-2001, Wainwright-2003, Cooper-deGivry-Schiex-2007]. We follow [Schlesinger-76] for its particular simplicity.

Primal LP

Joint states as probability distributions

Each hyperedge $A \in E$ is assigned a function $\mu_A \colon X_A \to \mathbb{R}$ that satisfies:

$$\begin{array}{lll} \text{(non-negativity)} & \mu_A(x_A) \geq 0 & A \in E;\ x_A \in X_A \\ \text{(normalisation)} & \displaystyle\sum_{x_A} \mu_A(x_A) = 1 & A \in E \end{array}$$

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Let $A = (1, 2, 3, 4)$ and $B = (1, 3)$. The marginalisation condition reads

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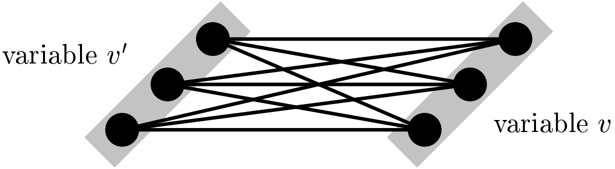
Note: To couple all pairs of overlapping distributions, we assume that E is closed under hyperedge intersection ($A, B \in E$ implies $A \cap B \in E$).

Example for $A = (v, v')$ and $B = (v)$:



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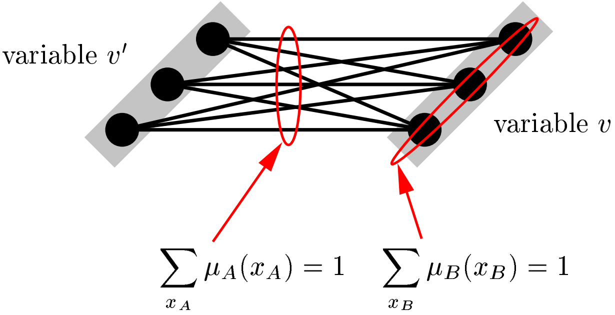

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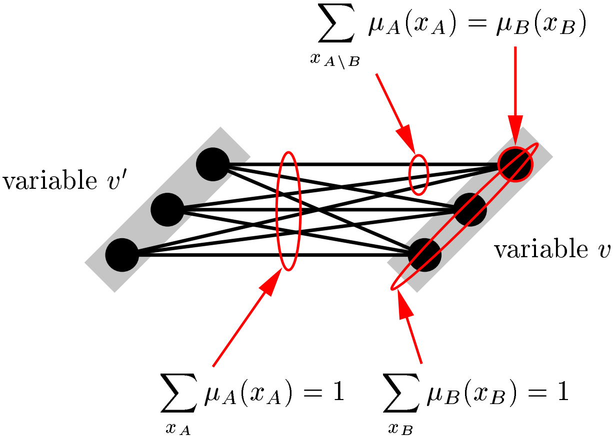

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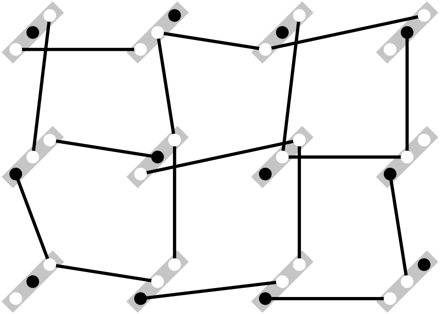
- Every distribution μ_A represents a single joint state.



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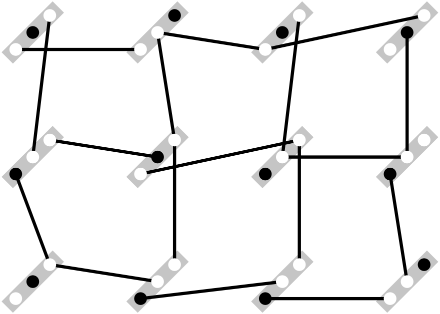

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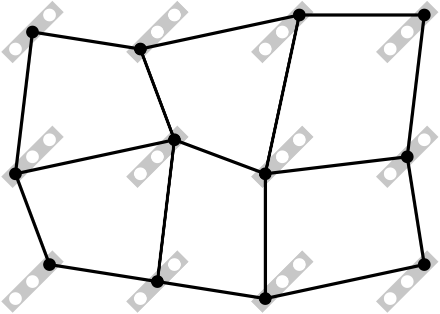

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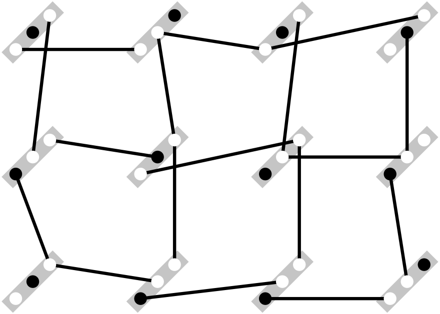
- The marginalisation constraint $\mu_B(x_B) = \sum_{x_{A \setminus B}} \mu_A(x_A)$ represents the fact that joint state μ_B is the restriction of joint state μ_A to variables $B \subset A$.
- The objective function is given by $f\,\mu = \sum_{A \in E} \sum_{x_A} f_A(x_A)\, \mu_A(x_A)$.



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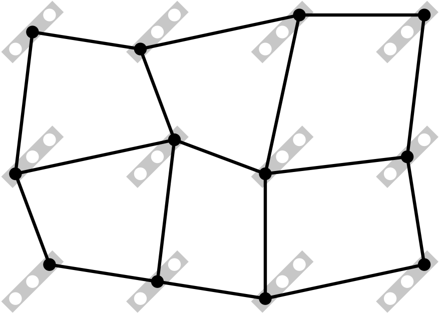

crisp μ satisfying also marginalisation

Theorem

Let $\binom{V}{1} \subseteq E$.
The WCSP optimum $\max\limits_{x_V} \sum\limits_{A \in E} f_A(x_A)$ *equals to the optimum of the integer LP*

$$
\begin{aligned}
& f\mu \to \max \\
\textit{subject to:} \quad & \mu_A(x_A) \in \{0,1\} && A \in E,\ x_A \in X_A \\
& \sum_{x_A} \mu_A(x_A) = 1 && A \in E \\
& \sum_{x_{A \setminus B}} \mu_A(x_A) = \mu_B(x_B) && A, B \in E,\ B \subset A,\ x_B \in X_B
\end{aligned}
$$

Relaxing $\mu_A(x_A) \in \{0,1\}$ to $\mu_A(x_A) \in [0,1]$ yields the primal LP.

Definition

- Networks f and f' are equivalent iff they yield the same objective function.
- A change of f to an equivalent network is an equivalent transformation.

An equivalent transformation is local iff it is applied to a triplet (A, B, x_B) with $B \subset A$ as follows:

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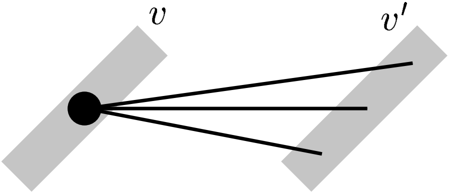

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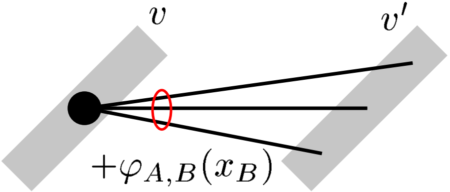

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- subtract the same constant from weight $f_B(x_B)$

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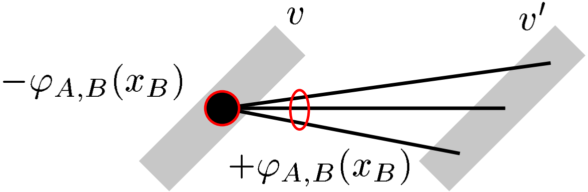

- Assign a constant $\varphi_{A,B}(x_B)$ to every such triplet (A, B, x_B) in the network. All these constants together are denoted by φ
- Applying local equivalent transformations on all triplets of a network f yields an equivalent network f^φ given by

$$f_A^\varphi(x_A) = f_A(x_A) + \sum_{B \subset A} \varphi_{A,B}(x_B) - \sum_{B \supset A} \varphi_{B,A}(x_A)$$

- Problems f^φ for all φ form an affine subspace of the space of all networks.
- For $f_A(x_A) > -\infty$, this subspace contains all networks equivalent with f.

Theorem (Upper bound on WCSP)

$$\underbrace{\max_{x_V} \sum_{A \in E} f_A(x_A)}_{WCSP\ optimum} \leq \sum_{A \in E} \max_{x_A} f_A(x_A)$$

The best upper bound is found by minimising it over equivalent networks:

$$\min_{\varphi} \sum_{A \in E} \max_{x_A} \Big[\underbrace{f_A(x_A) + \sum_{B \subset A} \varphi_{A,B}(x_B) - \sum_{B \supset A} \varphi_{B,A}(x_A)}_{f_A^{\varphi}(x_A)} \Big]$$

This can be written as a linear program

$$\begin{aligned} & \sum_{A \in E} \psi_A \to \min_{\varphi, \psi} \\ \text{subject to:} \quad & f_A^{\varphi}(x_A) \leq \psi_A \qquad A \in E,\ x_A \in X_A \end{aligned}$$

When is the upper bound tight?

Definition

Joint state x_A of hyperedge $A \in E$ is called $\begin{cases} \text{active} & \text{if} \quad f_A(x_A) = \max_{y_A} f_A(y_A) \\ \text{inactive} & \text{if} \quad f_A(x_A) < \max_{y_A} f_A(y_A) \end{cases}$

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Theorem

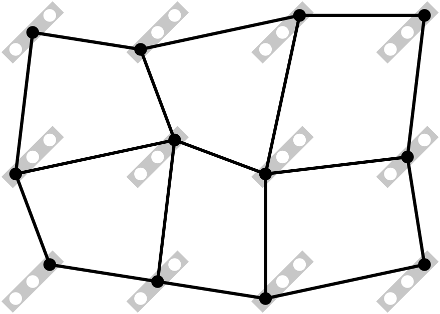
The upper bound is tight iff the (crisp) CSP formed by the active joint states is satisfiable.

When is the upper bound tight?

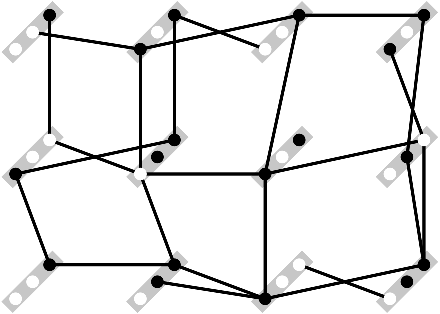
objective function for joint state x_V

upper bound

$$\sum_{A \in E} f_A(x_A) \quad \leq \quad \sum_{A \in E} \max_{x_A} f_A(x_A)$$



joint states given by x_V



active joint states

When is the upper bound tight?

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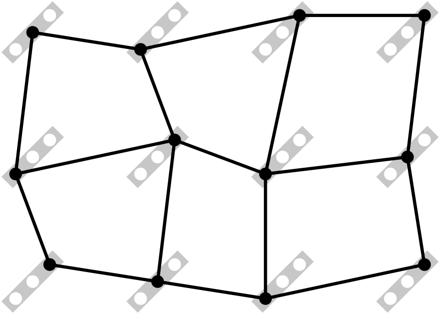

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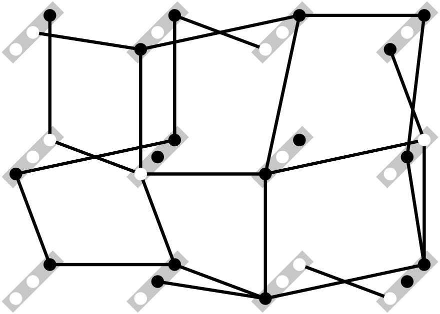


active joint states

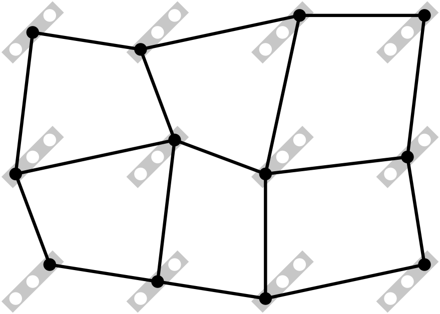
Do they form a satisfiable CSP?

When is the upper bound tight?

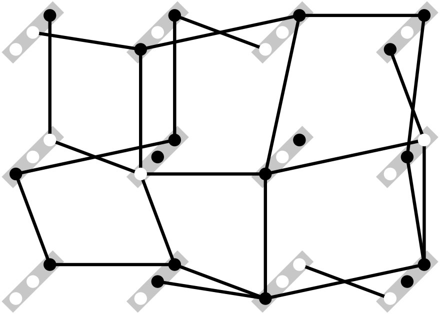
objective function for joint state x_V

upper bound

$$\sum_{A \in E} f_A(x_A) \quad < \quad \sum_{A \in E} \max_{x_A} f_A(x_A)$$



joint states given by x_V



active joint states

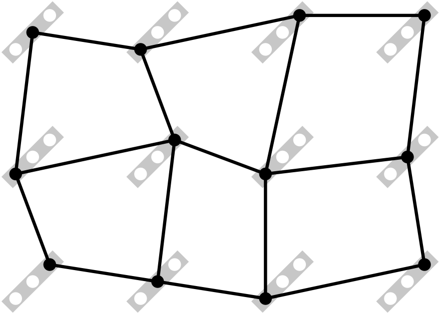
Do they form a satisfiable CSP?

No!

When is the upper bound tight?

objective function for joint state x_V upper bound

$$\sum_{A \in E} f_A(x_A) \quad \leq \quad \sum_{A \in E} \max_{x_A} f_A(x_A)$$



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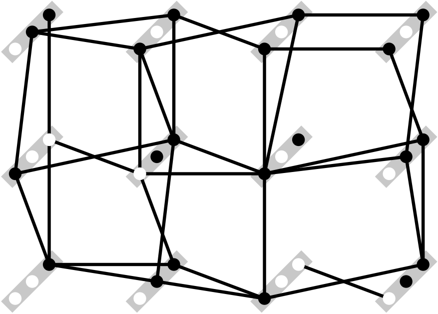

active joint states

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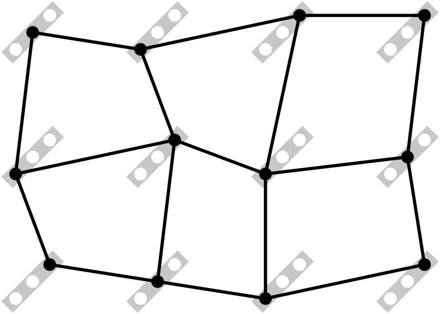
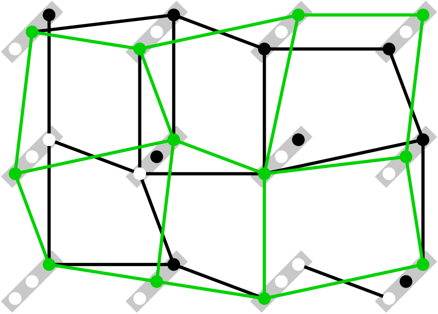


joint states given by x_V

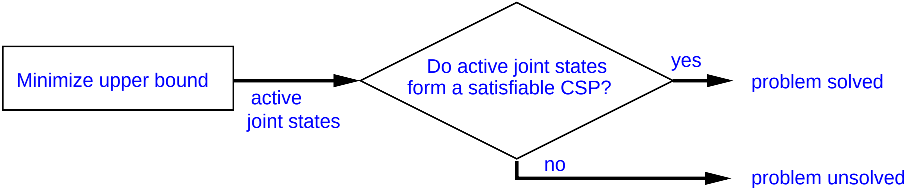


active joint states

Do they form a satisfiable CSP?

Yes!

The approach summarised



The instances that are solved exactly form a large, highly non-trivial tractable WCSP subclass.

LP duality

$$f\mu \to \max_{\mu} \qquad\qquad \sum_{A\in E} \psi_A \to \min_{\varphi,\psi}$$

$$\sum_{x_{A\setminus B}} \mu_A(x_A) = \mu_B(x_B) \qquad\qquad \varphi_{A,B}(x_B) \lessgtr 0 \qquad \begin{cases} A, B \in E \\ B \subset A \\ x_B \in X_B \end{cases}$$

$$\sum_{x_A} \mu_A(x_A) = 1 \qquad\qquad \psi_A \lessgtr 0 \qquad A \in E$$

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$$f\mu \to \max_{\mu} \qquad \psi 1 \to \min_{\varphi,\psi}$$

$$M\mu = 0 \qquad \varphi \lesseqgtr 0$$

$$N\mu = 1 \qquad \psi \lesseqgtr 0$$

$$\mu \geq 0 \qquad \varphi M + \psi N \geq f$$

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$$\mu_A(x_A) \geq 0 \qquad\qquad \underbrace{f_A(x_A) + \sum_{B\subset A}\varphi_{A,B}(x_B) - \sum_{B\supset A}\varphi_{B,A}(x_A)}_{f_A^{\varphi}(x_A)} \leq \psi_A \qquad \begin{cases} A\in E \\ x_A\in X_A \end{cases}$$

$$f\mu \to \max_{\mu} \qquad\qquad \psi 1 \to \min_{\varphi,\psi}$$

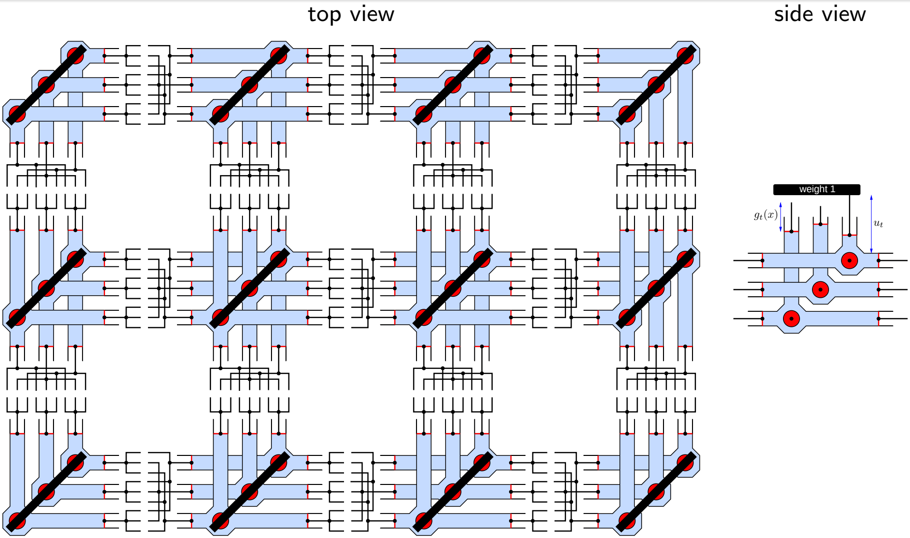
$$M\mu = 0 \qquad\qquad \varphi \lessgtr 0$$

$$N\mu = 1 \qquad\qquad \psi \lessgtr 0$$

$$\mu \geq 0 \qquad\qquad \varphi M + \psi N \geq f$$

primal program **dual program**

Hydraulic model of the LP pair [Schlesinger-Kovalevsky-1970's]



- $\mu_A(x_A)$ correspond to forces in forks and presures in tanks.
- $\varphi_{A,B}(x_A)$ correspond to displacements of the pistons.
- The upper bound corresnond to potential energy of the device.

LP duality theorems

$$\begin{aligned} f\mu &\to \max \\ M\mu &= 0 \\ N\mu &= 1 \\ \mu &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} \psi 1 &\to \min \\ \varphi &\lesseqgtr 0 \\ \psi &\lesseqgtr 0 \\ \varphi M + \psi N &\geq f \end{aligned}$$

Feasible μ and (φ, ψ) satisfy:

weak duality: $f\mu \leq \psi 1$

strong duality: μ and (φ, ψ) are optimal iff $f\mu = \psi 1$

complementary slackness: μ and (φ, ψ) are optimal iff $(\varphi M + \psi N - f)\mu = 0$

Complementary slackness in WCSP

Expression $(\varphi M + \psi N - f)\mu = 0$ reads

$$\forall A \in E,\, x_A \in X_A : \quad \Big[\max_{y_A} f_A^\varphi(y_A) - f_A^\varphi(x_A)\Big]\mu_A(x_A) = 0$$

which means:

Theorem (Complementary slackness in WCSP)

Let μ be primal-feasible. The primal and dual LPs are simultaneously optimal iff every joint state x_A of every hyperedge $A \in E$ satisfies the implication

$$(A, x_A) \textit{ is inactive} \quad \Longrightarrow \quad \mu_A(x_A) = 0$$

Definition

The CSP formed by active joint states is relaxed-satisfiable iff there exists a feasible μ for which the above implication holds.

Theorem

A network f has the least upper bound iff the CSP formed by active joint states is relaxed-satisfiable.

A crisp CSP represented by $\sigma\colon T(E) \to \{0,1\}$

is relaxed satisfiable

iff there exists $\mu \le \sigma$ satisfying

$$\mu_A(x_A) \ge 0$$
$$\sum_{x_A} \mu_A(x_A) = 1$$
$$\sum_{x_{A\setminus B}} \mu_A(x_A) = \mu_B(x_B)$$

Three levels of consistency of crisp CSP

A crisp CSP represented by $\sigma\colon T(E) \to \{0,1\}$

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Three levels of consistency of crisp CSP

A crisp CSP represented by $\sigma\colon T(E) \to \{0,1\}$

is satisfiable	is relaxed satisfiable	has a non-empty pairwise consistency closure

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$\mu_A(x_A) \in \{0,1\}$	$\mu_A(x_A) \ge 0$	$\mu_A(x_A) \in \{0,1\}$
$\sum_{x_A} \mu_A(x_A) = 1$	$\sum_{x_A} \mu_A(x_A) = 1$	$\max_{x_A} \mu_A(x_A) = 1$
$\sum_{x_{A \setminus B}} \mu_A(x_A) = \mu_B(x_B)$	$\sum_{x_{A \setminus B}} \mu_A(x_A) = \mu_B(x_B)$	$\max_{x_{A \setminus B}} \mu_A(x_A) = \mu_B(x_B)$

Three levels of consistency of crisp CSP

A crisp CSP represented by $\sigma\colon T(E) \to \{0,1\}$

is satisfiable $\implies$ is relaxed satisfiable $\implies$ has a non-empty pairwise consistency closure

iff there exists $\mu \leq \sigma$ satisfying

$$\mu_A(x_A) \in \{0,1\} \qquad \sum_{x_A} \mu_A(x_A) = 1 \qquad \sum_{x_{A\setminus B}} \mu_A(x_A) = \mu_B(x_B)$$

$$\mu_A(x_A) \geq 0 \qquad \sum_{x_A} \mu_A(x_A) = 1 \qquad \sum_{x_{A\setminus B}} \mu_A(x_A) = \mu_B(x_B)$$

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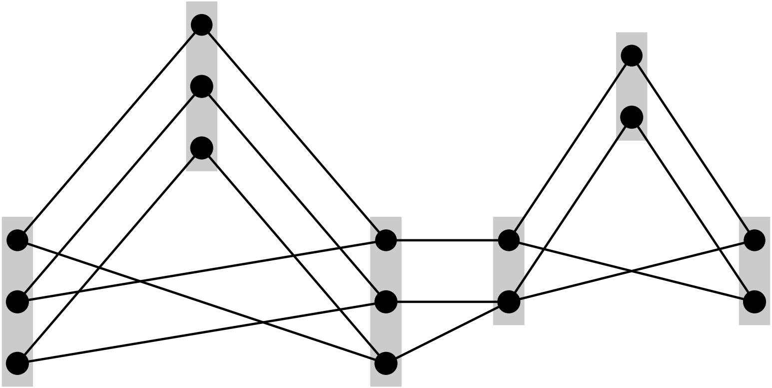
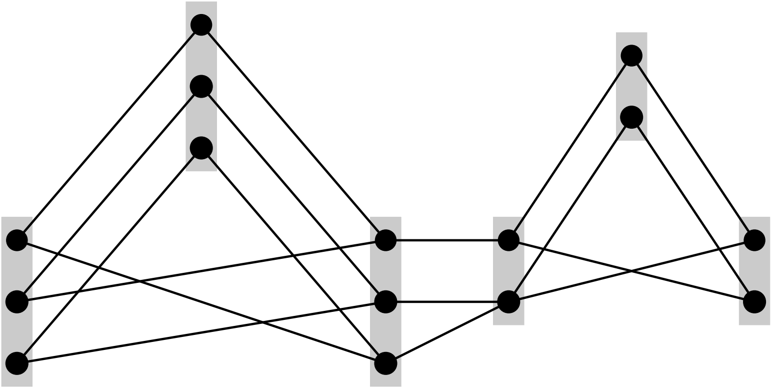
upper bound is tight

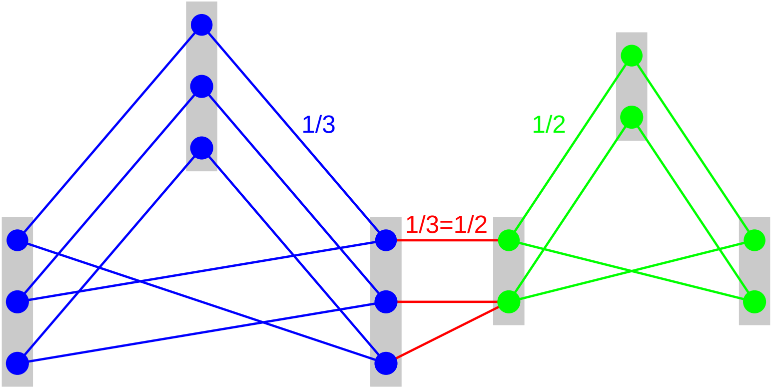
upper bound is minimal



- This is an arc consistent CSP.



- This is an arc consistent CSP.
- But there exists no μ on allowed joint states satisfying non-negativity, normalisation and marginalisation condition.



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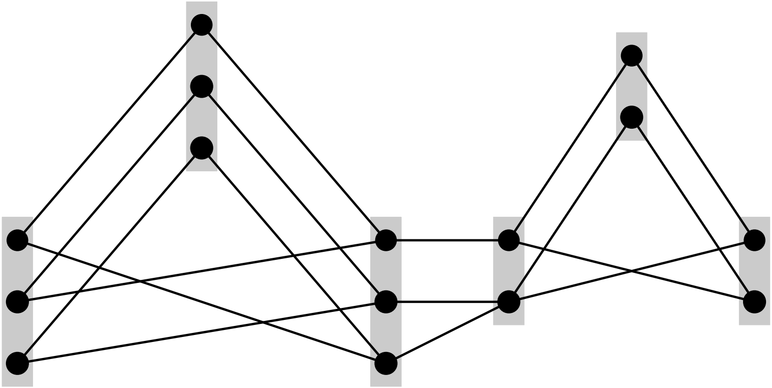
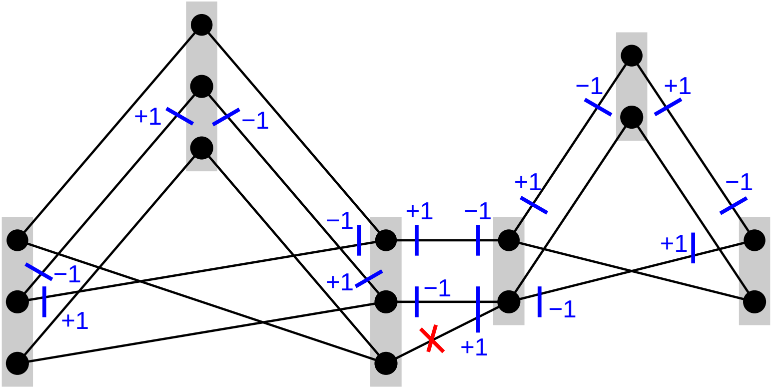


There exists an equivalent transformation of f that makes arc consistent closure empty. Hence, the upper bound can be decreased.



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Part 2

Algorithms to Decrease the Upper Bound

The dual LP more suitable to solve than the primal LP because

- it has fewer variables,
- all the optimal solutions are encoded in active joint states.

We need to solve

$$\min_{\varphi} \sum_{A \in E} \max_{x_A} \Bigg[\underbrace{f_A(x_A) + \sum_{B \subset A} \varphi_{A,B}(x_B) - \sum_{B \supset A} \varphi_{B,A}(x_A)}_{f_A^{\varphi}(x_A)} \Bigg]$$

which is an unconstrained minimisation problem
with convex and nonsmooth (piecewise linear) objective function.

- Restriction on the algorithm: space complexity must be linear in the number of variables $\varphi_{A,B}(x_B)$.

- Algorithms that make active joint states pairwise/arc consistent
(more efficient, but not guaranteed to find global minimum of upper bound)

- Algorithms that find global minimum of the upper bound
(typically less efficient)

Survey of existing approaches

- Algorithms that make active joint states pairwise/arc consistent
  (more efficient, but not guaranteed to find global minimum of upper bound)
  - Max-sum diffusion [Koval-Kovalevsky-1976, Flach-2000, Werner-2008]
  - Sequential Tree-reweighted Message Passing (TRW-S) [Wainwright-etal-2003, Kolmogorov-2005]
  - Augmenting DAG algorithm [Koval-Schlesinger-1976]
    virtual arc consistency algorithm [Cooper-deGivry-Schiex-2006-8]

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  These algorithms yield exact solution for supermodular WCSPs!

- Algorithms that find global minimum of the upper bound
  (typically less efficient)

  - Subgradient descent [Schlesinger-Giginjak-2007, Komodakis-etal-2007]
  - Smoothing methods [Werner-2007, Johnson-etal-2007]

Max-sum diffusion repeats this simple operation:

- On a triplet (A, B, x_B) with $B \subset A$, apply the local equivalent transformation that makes satisfied the equality $\max_{x_{A \setminus B}} f_A(x_A) = f_B(x_B)$

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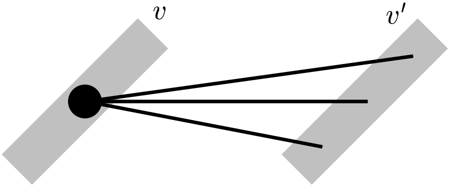
Example for $A = (v, v')$ and $B = (v)$:



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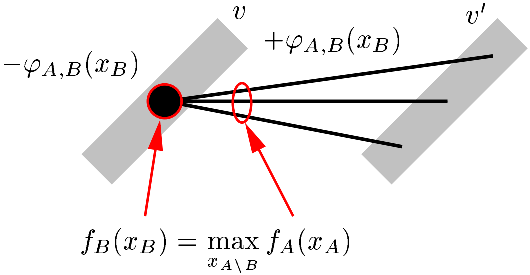


Algorithm: Max-sum diffusion

```
loop
   for (A, B) such that A ∈ E, B ∈ E, B ⊂ A do
      for x_B ∈ X_B do
         φ_{A,B}(x_B) ← φ_{A,B}(x_B) + [f_B^φ(x_B) − max_{x_{A\B}} f_A^φ(x_A)]/2
      end for
   end for
end loop
```

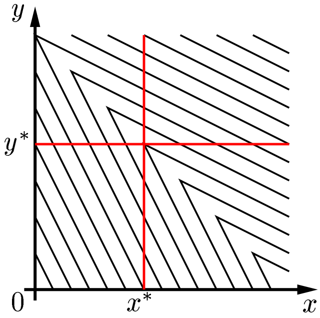
Properties of the algorithm:

- It monotonically decreases the upper bound.
- (Conjecture) It converges to a fixed point when $\max_{x_{A \setminus B}} f_A^{\varphi}(x_A) = f_B^{\varphi}(x_B)$ holds for all triplets (A, B, x_B) with $B \subset A$.
- At a fixed point, the CSP formed by active joint states is pairwise consistent.

Max-sum diffusion can be seen as a coordinate descent to decrease the upper bound:

- Minimise over a single free variable, keeping the other variables fixed.
- Iterate this for different free variables.

But coordinate descent is not guaranteed to find the global minimum of a convex nonsmooth function!



Point (x^*, y^*) is not a global minimum despite it is minimal separately in each coordinate.

Tree-reweighted Message Passing [Wainwright-etal-2003, Kolmogorov-2005]
Decomposing WCSP to tractable subproblems

- Let f^k denote a network with variables $V^k \subseteq V$ and hypergraph $E^k \subseteq 2^{V^k}$.
- Let $\{ f^k \mid k \in K \}$ be a collection of networks satisfying

$$f_A(x_A) = \sum_{k \mid A \in E^k} f_A^k(x_A) \qquad \forall A \in E,\ x_A \in X_A$$

- The objective function $F^k\colon X_{V^k} \to \bar{\mathbb{R}}$ of network f^k is $F^k(x_{V^k}) = \sum_{A \in E^k} f_A^k(x_A)$
- Swapping maximum and sum yields an upper bound on WCSP:

$$\max_{x_V} \sum_{A \in E} f_A(x_A) = \max_{x_V} \sum_k F^k(x_{V^k}) \le \sum_k \max_{x_{V^k}} F^k(x_{V^k})$$

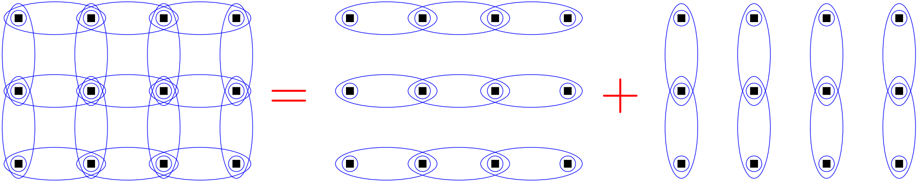
Minimising the upper bound

Minimise $\sum_k \max_{x_{V^k}} \sum_{A \in E^k} f_A^k(x_A)$ over collections $\{ f^k \mid k \in K \}$ subject to

$$f_A(x_A) = \sum_{k \mid A \in E^k} f_A^k(x_A)$$

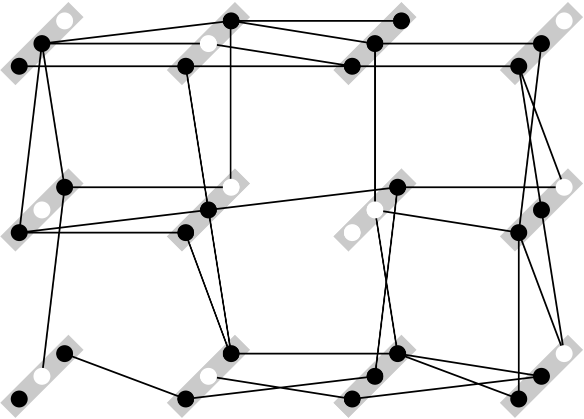
Tree-reweighted Message Passing [Wainwright-etal-2003, Kolmogorov-2005]

Example: Decomposing a grid graph to rows and columns

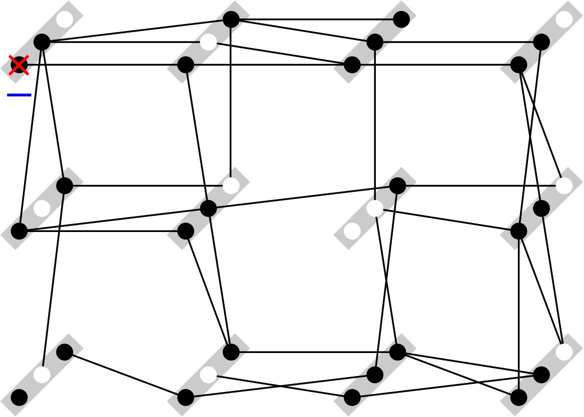


- Max-marginals of the row and column subproblems are iteratively equalised (like in max-sum diffusion) in the unary constraints.
- The row and column subproblems are solved efficiently by dynamic programming, in an incremental (hence efficient) way.

- Decomposing WCSP to subproblems is equivalent to approach by [Schlesinger-1976]. They can be translated to each other:
  - Expressing [Schlesinger-1976] by decomposition: Hypergraphs E^k are individual hyperedges $A \in E$.
  - Expressing decomposition by [Schlesinger-1976]: Each constraint f_A is itself a tractable WCSP, F^k.
- For binary problems, decomposing WCSP to trees yields (under mild assumptions) the same relaxation as the max-sum diffusion. But more efficient, especially on images.
- It is not straightforward to extend this efficiency to non-binary networks.



Run arc consistency algorithm on the active joint states. During this, remember pointers to causes of deletion. The pointers form an directed acyclic graph (DAG).



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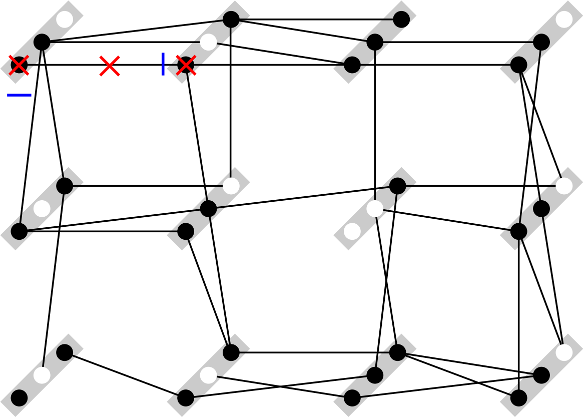

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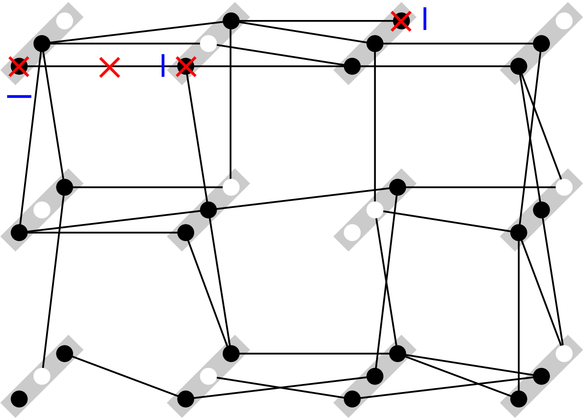

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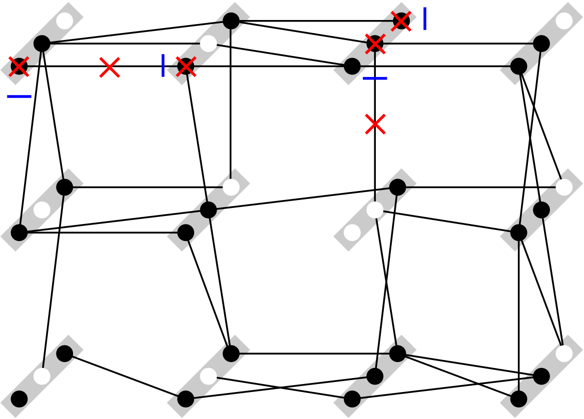

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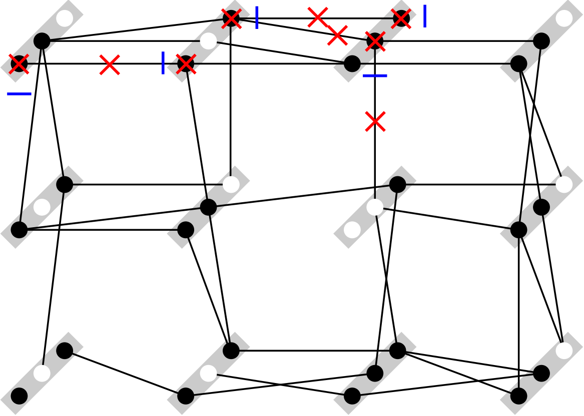

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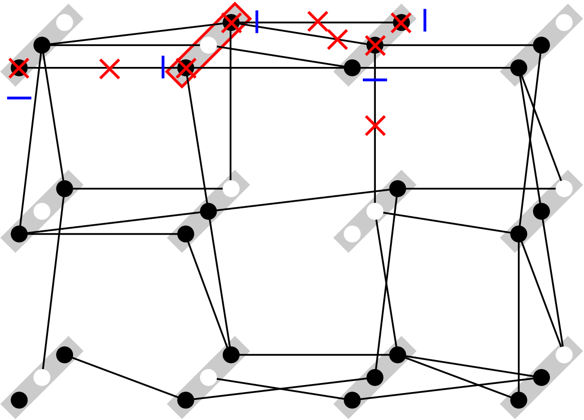
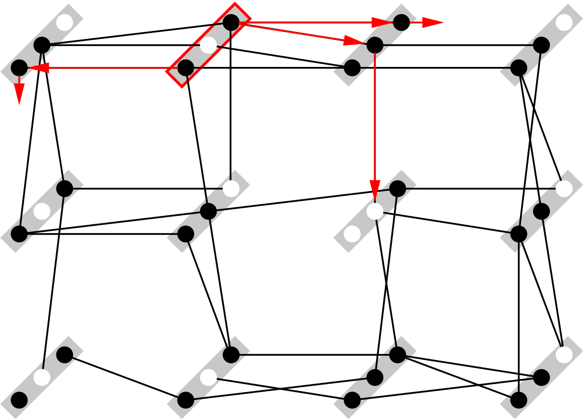
Augmenting DAG algorithm [Koval-Schlesinger-1976, Cooper-Schiex-2004]



Run arc consistency algorithm on the active joint states. During this, remember pointers to causes of deletion. The pointers form an directed acyclic graph (DAG).

- If all states in any variable are deleted, the upper bound cannot be minimal.



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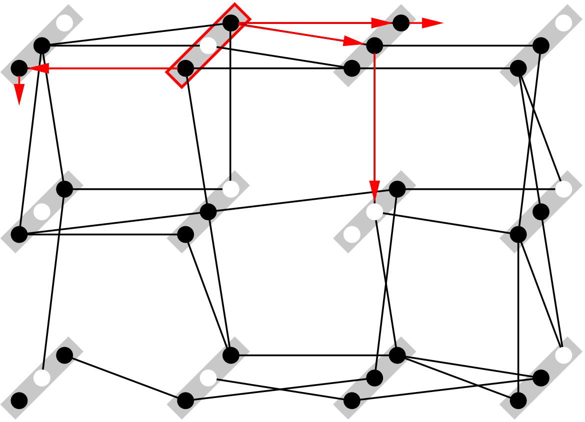
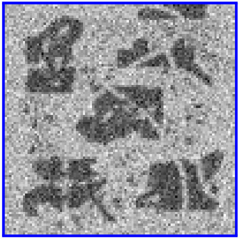


Run arc consistency algorithm on the active joint states. During this, remember pointers to causes of deletion. The pointers form an directed acyclic graph (DAG).

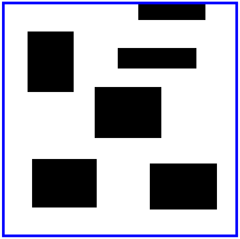
- If all states in any variable are deleted, the upper bound cannot be minimal. Then, it can be decreased by pushing weights along (a subgraph of) the DAG.
- If a non-empty arc consistency closure is found, halt.

Example WCSP: Syntactic image analysis

Find the image containing non-overlapping rectangles, nearest to input image!



input image



output image

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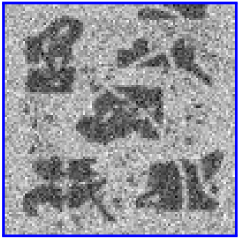

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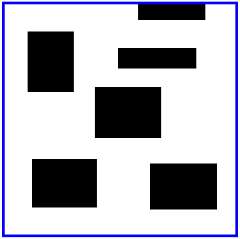

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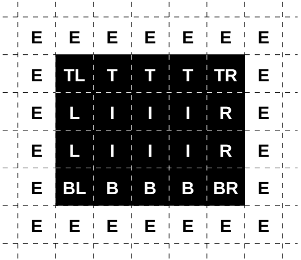
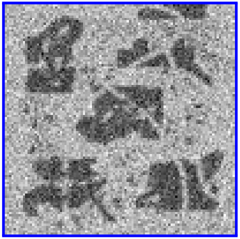
Example WCSP: Syntactic image analysis

Find the image containing non-overlapping rectangles, nearest to input image!

- Variables V are pixels, hypergraph E is the image grid.
- Variable domains $X_v = \{\,\mathrm{E}, \mathrm{I}, \mathrm{L}, \mathrm{R}, \mathrm{T}, \mathrm{B}, \mathrm{TL}, \mathrm{TR}, \mathrm{BL}, \mathrm{BR}\,\}$ are syntactic parts of a rectangle.



hidden states = syntactic parts



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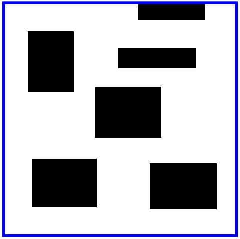

output image

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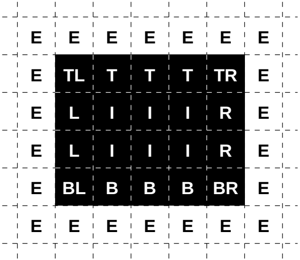
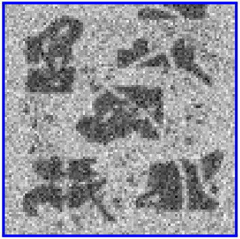
- Unary constraint $f_v(x_v)$ quantifies agreement of intensity of state x_v and intensity of input pixel v.



hidden states = syntactic parts
observed states = {black,white}



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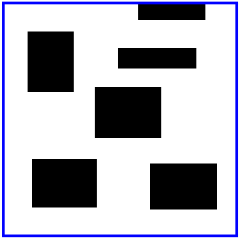

output image

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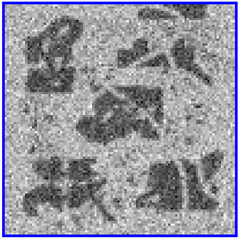
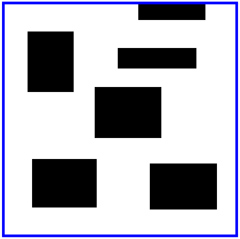
- Variables V are pixels, hypergraph E is the image grid.
- Variable domains $X_v = \{\, \text{E, I, L, R, T, B, TL, TR, BL, BR} \,\}$ are syntactic parts of a rectangle.
- Unary constraint $f_v(x_v)$ quantifies agreement of intensity of state x_v and intensity of input pixel v.
- Binary constraint $f_{vv'}(x_v, x_{v'})$ equals 0 if syntactic parts x_v and $x_{v'}$ are allowed to neighbor and $-\infty$ otherwise.

E	E	E	E	E	E	E
E	TL	T	T	T	TR	E
E	L	I	I	I	R	E
E	L	I	I	I	R	E
E	BL	B	B	B	BR	E
E	E	E	E	E	E	E

hidden states = syntactic parts
observed states = {black,white}



input image



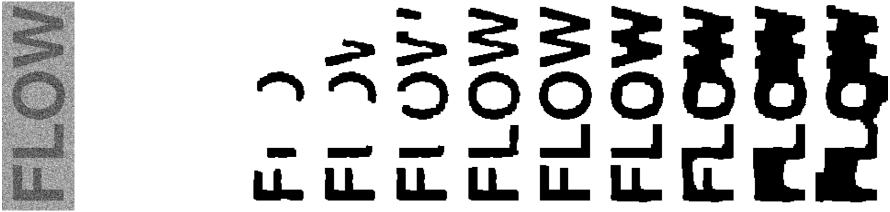
output image

Example: Binary WCSP with a global constraint [Werner-2008]

- $E = \binom{V}{1} \cup E' \cup \{V\}$ where $E' \subseteq \binom{V}{2}$, $X_v = \{\text{white}, \text{black}\}$.
- Unary constraint f_v quantifies agreement with intensity of input pixel v.
- Binary constraints $f_{vv'}$ penalise transition between black and white pixels.
- Global constraint $f_V(x_V)$ is: 0 if x_V contains n black pixels and $-\infty$ otherwise.

Interpretation: Find minimum *st*-cut in a graph such that the number of pixels in the first partition equals n (NP-hard).

- Max-sum diffusion enforces generalised arc consistency of active joint states.
- Equalising max-marginals between f_v and f_V seen as a soft global propagator.



input									
	n required:	2000	3000	4000	5000	5368	6000	7000	8000	9000
	n achieved:	2008	3004	4011	5006	5368	6004	7024	7982	9032

Supermodular problems

Let each domain X_v be totally ordered. A function f_A is supermodular if

$$f_A(x_A \wedge y_A) + f_A(x_A \vee y_A) \geq f_A(x_A) + f_A(y_A)$$

for any $x_A, y_A \in X_A$, where $\wedge$ ($\vee$) denotes the elementwise minimum (maximum).

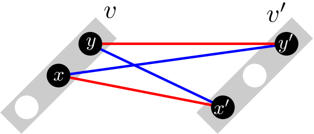
Theorem ([Schlesinger-Flach-00] for binary case, [Werner-2008] for non-binary case)

Let f_A be supermodular for each $A \in E$. Finding an equivalent network whose active joint states are generalised arc consistent solves the WCSP f exactly.

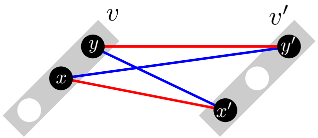
Proof

- Equivalent transformations preserve supermodularity.
- The maximisers of a supermodular function on a distributive lattice form a sublattice of this lattice [Topkis78]. Hence, the active joint states of each constraint form a lattice. Hence, all active joint states form a well-known tractable CSP [Jeavons-Cooper-95] (lattice CSP).
- Generalised arc consistency suffices for a lattice CSP to be satisfiable.

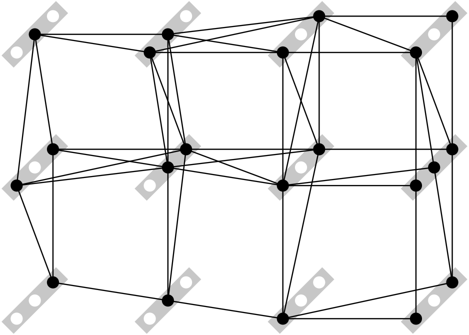
Before, [Cooper-2008] showed that the LP relaxation is tight for non-binary supermodular WCSPs. Our statement is stronger and the proof is simpler.

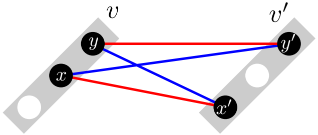


$$x \leq x', y \leq y' \quad \Longrightarrow \quad f_{vv'}(x, x') + f_{vv'}(y, y') \geq f_{vv'}(x, y') + f_{vv'}(y, x')$$

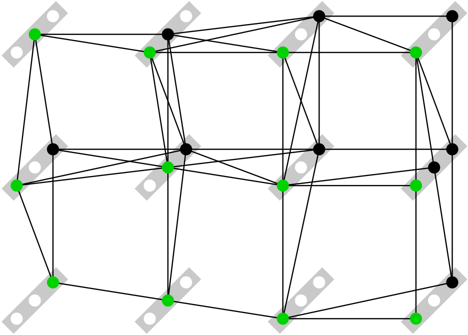


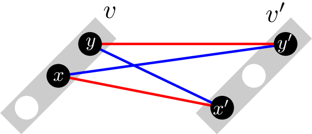
$$x \leq x', y \leq y' \quad \Longrightarrow \quad f_{vv'}(x, x') + f_{vv'}(y, y') \geq f_{vv'}(x, y') + f_{vv'}(y, x')$$



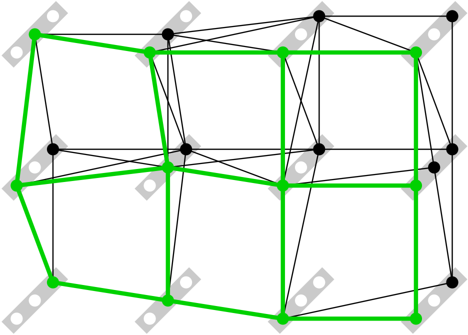


$$x \le x', y \le y' \quad \Longrightarrow \quad f_{vv'}(x, x') + f_{vv'}(y, y') \ge f_{vv'}(x, y') + f_{vv'}(y, x')$$





$$x \leq x', y \leq y' \quad \Longrightarrow \quad f_{vv'}(x, x') + f_{vv'}(y, y') \geq f_{vv'}(x, y') + f_{vv'}(y, x')$$



Subgradient descent

Well-known approach to minimise nonsmooth convex functions [Shor-1979].

Definition (Subgradient)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function. A vector $g(x_0) \in \mathbb{R}^n$ is a subgradient of f at point x_0 iff $g^\top (x - x_0) \le f(x) - f(x_0)$ for all $x \in \mathbb{R}^n$.

Theorem (Subgradient descent)

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $\{\alpha_i\}_{i=0}^{\infty}$ be a sequence of positive numbers that converge to zero but their partial sums converge to infinity. For any initial point x_0, the sequence

$$x_{i+1} = x_i - \alpha_i \, g(x_i)$$

converges to $\min_{x \in \mathbb{R}^n} f(x)$.

Apply gradient descent to minimising the WCSP upper bound [Schlesinger-Giginjak-2007, Komodakis-etal-2007]:

- Converges (non-monotonically) to the global minimum of the upper bound.
- Inefficient if applied to a decomposition to individual hyperedges.
- Efficient if applied to a decomposition to longer trees/chains.

Smoothing methods

- Let $f_\beta\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function for every $\beta > 0$.
- With increasing β, let f_β approach a nonsmooth convex function f_∞.
- The sequence $\min_{x\in\mathbb{R}^n} f_\beta(x)$ converges to $\min_{x\in\mathbb{R}^n} f_\infty(x)$.

$\beta \leftarrow 1$
loop
    Minimise f_β by coordinate descent.
    Increase β.
end loop

An example [Werner-2007, Johnson-etal-2007]:

- Approach crisp maximum by a sequence of soft maxima:

$$\max\{x, y\} = \lim_{\beta\to\infty} \frac{1}{\beta} \log\left(e^{\beta x} + e^{\beta y}\right)$$

  Minimising the smoothed upper bound leads to geometric programming.
- Not very practical (in my experience) because of the two nested loops in the algorithm.

Part 3

Higher Order Polyhedral Relaxations

Adding zero constraints

Let f be a network with hypergraph E. Let $A \notin E$. Add constraint $f_A \equiv 0$ to f.

Theorem

By adding a zero constraint to a network, the optimal value of the LP pair

- *never increases*
- *for some instances, strictly decreases.*

Proof, by primal argument

- The primal objective function is preserved.
- The new primal feasible set is a strict subset of the old one.

Proof, by dual argument

- The dual objective function (i.e., the upper bound) is preserved.
- New equivalent transformations are enabled and no existing ones disabled. Hence, the new dual feasible set is a strict superset of the old one.

The proofs are imprecise because the dimensions of the old and new feasible sets are different. Let us be more precise...

primal program	**dual program**	
$f\mu \to \max$	$\sum_{A\in E} \psi_A \to \min$	
$\mu_A(x_A) \geq 0$	$f_A(x_A) \leq \psi_A$	$A \subseteq V,\ x_A \in X_A$
$\sum_{x_A} \mu_A(x_A) = 1$	$\psi_A \lesseqgtr 0$	$A \subseteq V$

- Suppose that all possible (i.e., for all $A \subseteq V$) zero constraints are added but marginalisation conditions are omitted. The optimal value of the LP pair is

$$\sum_{A\subseteq V} \max_{x_A} f_A(x_A) = \sum_{A\in E} \max_{x_A} f_A(x_A)$$

Relaxation is given by marginalisation conditions

primal program	**dual program**	
$f\mu \to \max$	$\sum_{A\in E} \psi_A \to \min$	
$\mu_A(x_A) \geq 0$	$f_A^{\varphi}(x_A) \leq \psi_A$	$A \subseteq V,\ x_A \in X_A$
$\sum_{x_A} \mu_A(x_A) = 1$	$\psi_A \lesseqgtr 0$	$A \subseteq V$
$\sum_{x_{A\setminus B}} \mu_A(x_A) = \mu_B(x_B)$	$\varphi_{A,B}(x_B) \lesseqgtr 0$	$(A,B) \in J,\ x_B \in X_B$

- Suppose that all possible (i.e., for all $A \subseteq V$) zero constraints are added but marginalisation conditions are omitted. The optimal value of the LP pair is

$$\sum_{A\subseteq V} \max_{x_A} f_A(x_A) = \sum_{A\in E} \max_{x_A} f_A(x_A)$$

- Couple distribution pairs (μ_A, μ_B) for $(A, B) \in J$ by marginalisation, where

$$J \subseteq I(2^V) = \{\, (A,B) \mid B \subset A \subseteq V \,\} = \text{ the inclusion relation on } 2^V$$

Now, tightness of the relaxation is determined by J alone!
Zero constraints not participating in J are superfluous.

The same graphically...

- The set 2^V of all subsets (except $\emptyset$) of $V = (1, 2, 3, 4)$.

1234

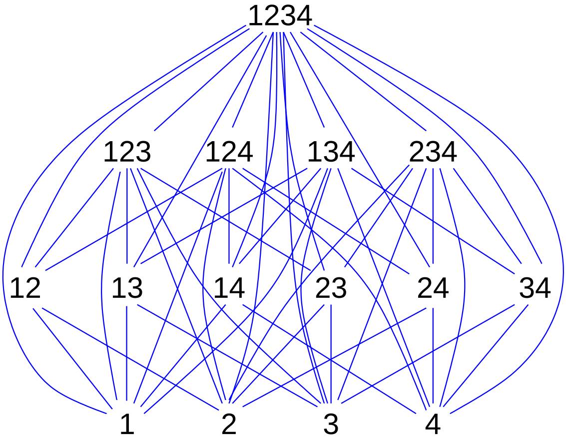
123 124 134 234

12 13 14 23 24 34

1 2 3 4

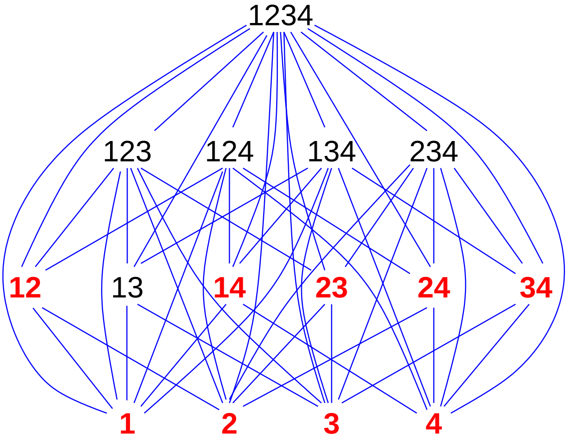
The same graphically...

- The set 2^V of all subsets (except $\emptyset$) of $V = (1, 2, 3, 4)$.
- Inclusion relation $I(2^V)$.

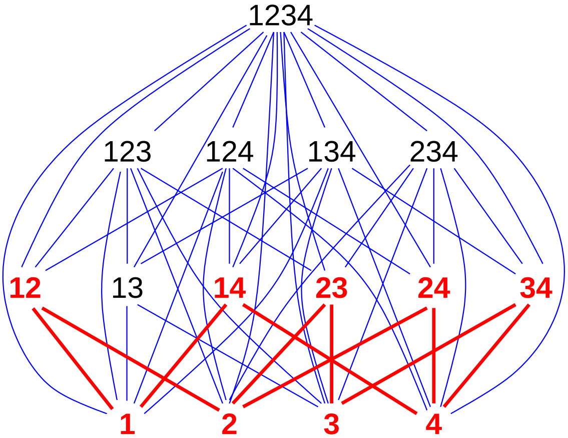


The same graphically...

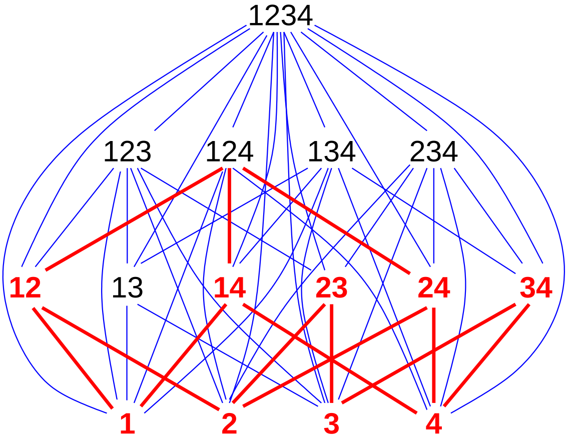
- The set 2^V of all subsets (except $\emptyset$) of $V = (1, 2, 3, 4)$.
- Inclusion relation $I(2^V)$.
- Non-zero constraints, given by hypergraph E.



- The set 2^V of all subsets (except $\emptyset$) of $V = (1, 2, 3, 4)$.
- Inclusion relation $I(2^V)$.
- Non-zero constraints, given by hypergraph E.
- Imposing marginalisations $J = I(E)$ yields the ordinary LP relaxation.



- The set 2^V of all subsets (except $\emptyset$) of $V = (1, 2, 3, 4)$.
- Inclusion relation $I(2^V)$.
- Non-zero constraints, given by hypergraph E.
- Imposing marginalisations $J = I(E)$ yields the ordinary LP relaxation.
- Adding these three pairs (A, B) to J tightens the relaxation.
  This requires adding zero constraint $f_{124} \equiv 0$.



Hierarchy of polyhedral relaxations

$J_1 \subseteq J_2$ implies that relaxation J_1 is not tighter than J_2. Therefore:

Theorem

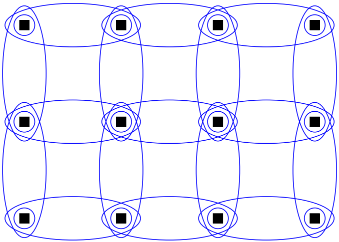
All possible sets $J \subseteq I(2^V)$ form a hierarchy of relaxations, partially ordered by the inclusion relation on $I(2^V)$.

In particular:

$J = \emptyset$	the weakest relaxation (the sum of independent maxima of all the constraints).
$J = I(E)$	the ordinary LP relaxation [Schlesinger-76, other works...]
$J = I(2^V)$	the exact solution (the same as adding the single zero constraint $f_V \equiv 0$)

Example: Adding zero 4-cycle constraints to binary problems

- Randomly draw instances of a binary problem from an instance type.
- Count the number of instances solved exactly by an LP relaxation.
- Two relaxations tested:
  - Plain LP relaxation without zero constraints

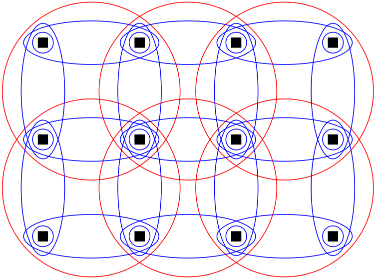


without zero constraints

type	image side	$\lvert X_v \rvert$	r_{plain}
random	15	5	0.01
random	25	3	0.00
random	100	3	0.00
Potts	15	5	0.79
Potts	25	5	0.48
Potts	100	5	0.00
lines	10	4	0.72
lines	25	4	0.00
curve	10	9	0.17
curve	15	9	0.00
curve	25	9	0.00
Pi	15	5	0.00

Example: Adding zero 4-cycle constraints to binary problems

- Randomly draw instances of a binary problem from an instance type.
- Count the number of instances solved exactly by an LP relaxation.
- Two relaxations tested:
  - Plain LP relaxation without zero constraints
  - LP relaxation augmented by 4-ary zero constraints on neighboring variables.



with zero constraints

type	image side	$\|X_v\|$	r_{plain}	r_{4cycle}
random	15	5	0.01	1.00
random	25	3	0.00	0.98
random	100	3	0.00	0.72
Potts	15	5	0.79	0.99
Potts	25	5	0.48	0.98
Potts	100	5	0.00	0.81
lines	10	4	0.72	0.88
lines	25	4	0.00	0.00
curve	10	9	0.17	0.65
curve	15	9	0.00	0.24
curve	25	9	0.00	0.00
Pi	15	5	0.00	0.82

Consider the LP relaxation $\max \left\{ f\mu \;\middle|\; \mu \in P \right\}$ of ILP $\max \left\{ f\mu \;\middle|\; \mu \in P \cap \mathbb{Z}^n \right\}$.

Consider the LP relaxation $\max \{ f\mu \mid \mu \in P \}$ of ILP $\max \{ f\mu \mid \mu \in P \cap \mathbb{Z}^n \}$.

Cutting plane algorithm for general ILP (in primal space)

Cutting plane algorithm

Consider the LP relaxation $\max\left\{ f\mu \;\middle|\; \mu \in P \right\}$ of ILP $\max\left\{ f\mu \;\middle|\; \mu \in P \cap \mathbb{Z}^n \right\}$.

Cutting plane algorithm for general ILP (in primal space)

1: $P' \leftarrow P$

Cutting plane algorithm

Consider the LP relaxation $\max \{ f\mu \mid \mu \in P \}$ of ILP $\max \{ f\mu \mid \mu \in P \cap \mathbb{Z}^n \}$.

Cutting plane algorithm for general ILP (in primal space)

$P' \leftarrow P$
loop
  Find a maximiser μ^* of $\max \{ f\mu \mid \mu \in P' \}$.
end loop

Consider the LP relaxation $\max\left\{ f\mu \;\middle|\; \mu \in P \right\}$ of ILP $\max\left\{ f\mu \;\middle|\; \mu \in P \cap \mathbb{Z}^n \right\}$.

Cutting plane algorithm for general ILP (in primal space)

$P' \leftarrow P$
loop
    Find a maximiser μ^* of $\max\left\{ f\mu \;\middle|\; \mu \in P' \right\}$.
    Find a half-space H such that $\mu^* \notin H \supseteq P \cap \mathbb{Z}^n$. If none exists, halt.

end loop

Cutting plane algorithm

Consider the LP relaxation $\max\left\{ f\mu \mid \mu \in P \right\}$ of ILP $\max\left\{ f\mu \mid \mu \in P \cap \mathbb{Z}^n \right\}$.

Cutting plane algorithm for general ILP (in primal space)

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    $P' \leftarrow P' \cap H$
end loop

Consider the LP relaxation $\max\left\{ f\mu \,\middle|\, \mu \in P \right\}$ of ILP $\max\left\{ f\mu \,\middle|\, \mu \in P \cap \mathbb{Z}^n \right\}$.

Cutting plane algorithm for general ILP (in primal space)

Cutting plane algorithm for WCSP (in dual space)

Cutting plane algorithm

Consider the LP relaxation $\max\{\, f\mu \mid \mu \in P \,\}$ of ILP $\max\{\, f\mu \mid \mu \in P \cap \mathbb{Z}^n \,\}$.

Cutting plane algorithm for general ILP (in primal space)

$P' \leftarrow P$
loop
    Find a maximiser μ^* of $\max\{\, f\mu \mid \mu \in P' \,\}$.
end loop

Cutting plane algorithm for WCSP (in dual space)

loop
    Find the equivalent network with the least upper bound.
end loop

Cutting plane algorithm

Consider the LP relaxation $\max\{\, f\mu \mid \mu \in P \,\}$ of ILP $\max\{\, f\mu \mid \mu \in P \cap \mathbb{Z}^n \,\}$.

Cutting plane algorithm for general ILP (in primal space)

$P' \leftarrow P$
loop
    Find a maximiser μ^* of $\max\{\, f\mu \mid \mu \in P' \,\}$.
    Find a half-space H such that $\mu^* \notin H \supseteq P \cap \mathbb{Z}^n$. If none exists, halt.

end loop

Cutting plane algorithm for WCSP (in dual space)

loop
    Find the equivalent network with the least upper bound.
    Find $A \notin E$ such that the CSP formed by active joint states restricted on hypergraph $E \cap 2^A$ is unsatisfiable. If none exists, halt.

end loop

Cutting plane algorithm

Consider the LP relaxation $\max\{\, f\mu \mid \mu \in P \,\}$ of ILP $\max\{\, f\mu \mid \mu \in P \cap \mathbb{Z}^n \,\}$.

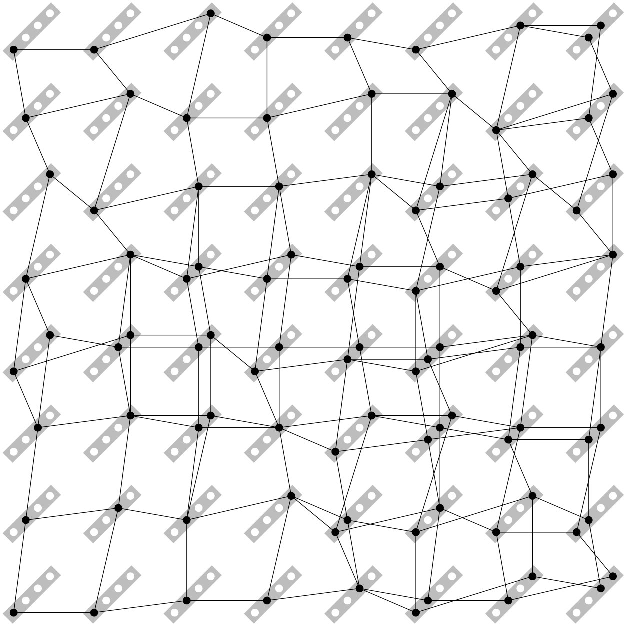
Cutting plane algorithm for general ILP (in primal space)

```
P' ← P
loop
    Find a maximiser μ* of max{ fμ | μ ∈ P' }.
    Find a half-space H such that μ* ∉ H ⊇ P ∩ ℤⁿ. If none exists, halt.
    P' ← P' ∩ H
end loop
```

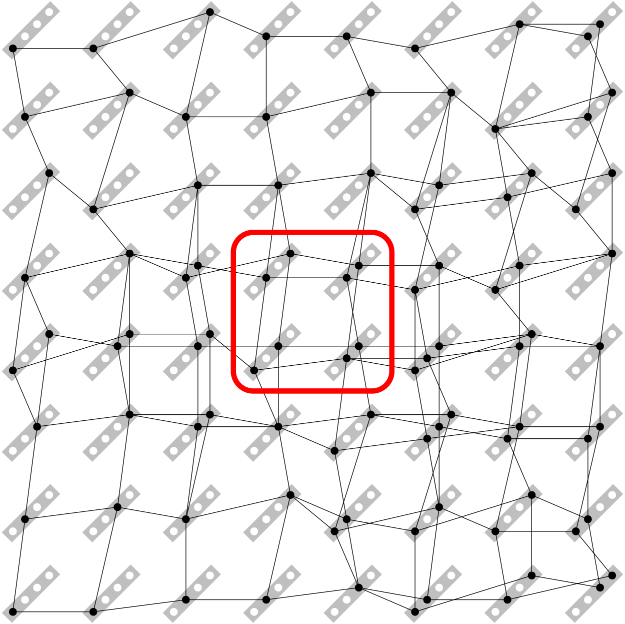
Cutting plane algorithm for WCSP (in dual space)

```
loop
    Find the equivalent network with the least upper bound.
    Find A ∉ E such that the CSP formed by active joint states restricted on
    hypergraph E ∩ 2^A is unsatisfiable. If none exists, halt.
    f_A ← 0;  E ← E ∪ {A}
end loop
```

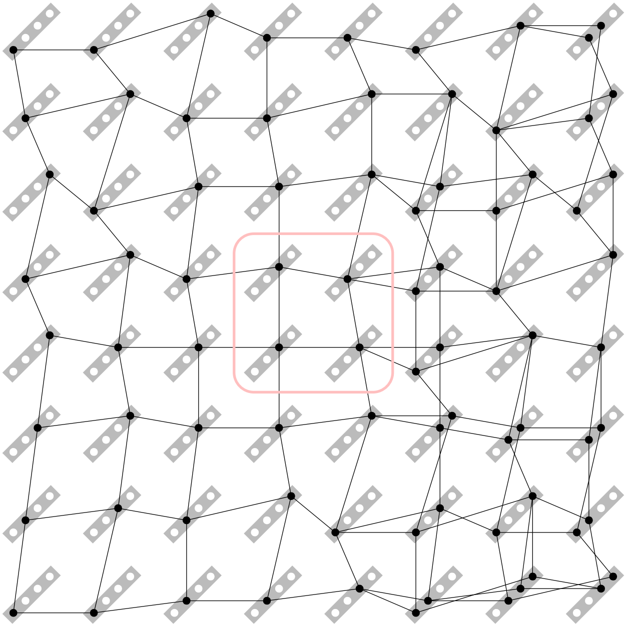
Example run of cutting plane algorithm

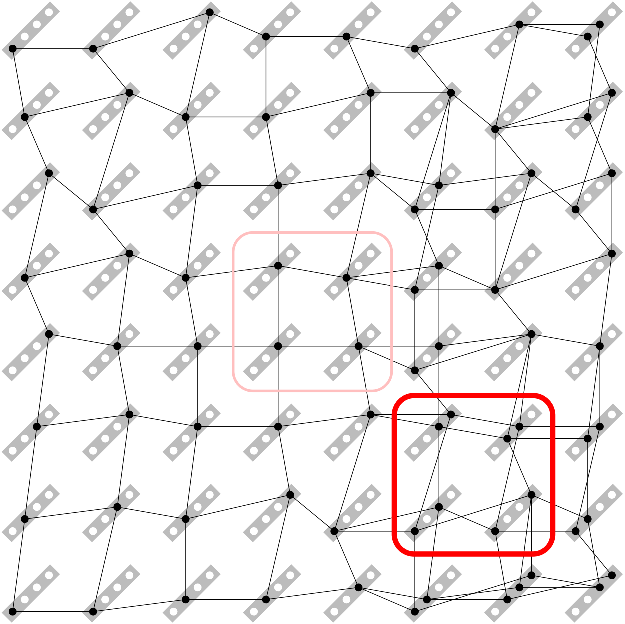


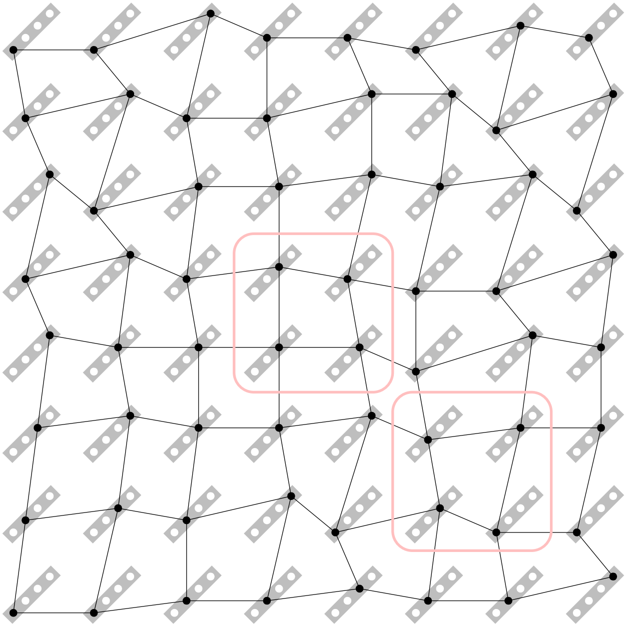
Example run of cutting plane algorithm



Example run of cutting plane algorithm







Integral hull of LP relaxation

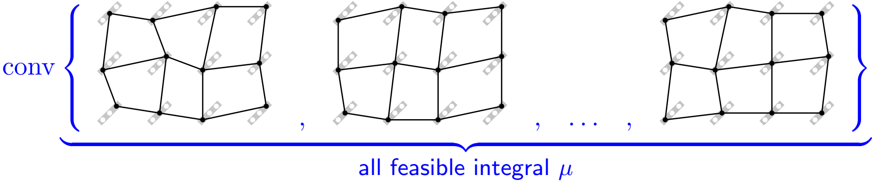
LP relaxation is a common approach to combinatorial optimisation problems:

$$\max\{\, f\mu \mid \mu \in \underbrace{\mathrm{conv}(P \cap \mathbb{Z}^n)}_{\substack{\text{integral hull}\\ \text{of } P}} \,\} = \overbrace{\max\{\, f\mu \mid \mu \in P \cap \mathbb{Z}^n \,\}}^{\text{integer LP}} \leq \overbrace{\max\{\, f\mu \mid \mu \in P \,\}}^{\text{LP relaxation}}$$

- P is a convex polyhedron with a tractable number of facets.
- For NP-hard ILPs, $\mathrm{conv}(P \cap \mathbb{Z}^n)$ has much more facets than P and they cannot be described in a 'short' way. Facets of $\mathrm{conv}(P \cap \mathbb{Z}^n)$ that are not facets of P are good cutting planes.
- All vertices of $\mathrm{conv}(P \cap \mathbb{Z}^n)$ are integral while P may have also fractional vertices.

Integral hull of WCSP = marginal polytope

Integral hull of WCSP is the set



Definition (Marginal polytope [Wainwright-etal-2003])

Marginal polytope (associated with hypergraph $E \subseteq 2^V$ and domains $\{X_v\}$) is a set of mappings $\mu\colon T(E) \to [0,1]$ defined as follows: μ belongs to the marginal polytope iff there exists a function $\mu_V\colon X_V \to [0,1]$ such that $\sum_{x_V} \mu_V(x_V) = 1$ and

$$\forall A \in E,\, x_A \in X_A : \qquad \sum_{x_{V \setminus A}} \mu_V(x_V) = \mu_A(x_A)$$

That is, μ is a set of valid marginals of some global distribution μ_V.

Theorem

Marginal polytope is the integral hull of WCSP.